# Developments of a Generalized Theory of Jet Noise

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**Following a generalized theory of supersonic jet noise, explicit analytical expressions for identifying jet noise mechanisms and numerical calculations of high-speed jet noise characteristics have been obtained. This new theory is uniformly valid in the transonic and supersonic speed ranges, and it includes implicitly refraction and convection of sound through the jet flow. Given results include the transition from $U^8$-law to the $U^3$-law, differences in directivity for high and low frequencies, sound source distribution, noise spectrum, and the coupled effects of refraction and convection. The implications of these results and their correspondences with results in the existing literature are discussed.**

## Introduction

IN 1960, Phillips[1] derived a convected wave equation for studying aerodynamical noise generating processes in high-speed turbulent flows. In this equation, mean flow properties such as local velocity and local speed of sound are explicitly represented in the mathematical descriptions. By restricting the mean flow to a parallel shear layer, a special solution to the convected wave equation was given in Ref. 1. This solution deals with Mach number radiations from hypersonic shear layers. Recently, Pao[2,3] obtained a complete set of solutions to the Phillips equation which is uniformly valid in the subsonic, the transonic, and the supersonic convection speed ranges. Since the Phillips wave equation is inherently a very accurate description of the dynamics of wave radiation from a parallel shear flow, new insight into jet noise generating mechanisms can be derived from these analytical solutions.

In this paper, the prime purpose is to derive from the theory some key properties of jet noise, such as sound power dependence, spectrum, sound source strength distribution, directivity, and temperature effects; and to relate the results of this generalized theory to previous results in the literature. Regarding the conventional approach of studying jet noise, questions are often raised concerning the effects of refraction, convection, compressibility of the gas flow, or the validity of the fundamentals. Some answers to these questions are found in this generalized approach. Furthermore, it was found that the Lighthill theory is closely related to the generalized theory. Their subtle differences, as well as the similarities, will also be discussed.

## Formulation

The convected wave equation, which is derived from the momentum equation, the continuity equation, and the equation of state for a perfect gas, can be written as:

$$\frac{D^2}{Dt^2}\log\left(\frac{p}{p_0}\right)-\frac{\partial}{\partial x_i}\left\{c^2\frac{\partial}{\partial x_i}\log\left(\frac{p}{p_0}\right)\right\}=\gamma\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}+\gamma\frac{D}{Dt}\left(\frac{1}{c_p}\frac{DS}{Dt}\right)-\gamma\frac{\partial}{\partial x_i}\left\{\frac{1}{\rho}\frac{\partial}{\partial x_j}\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\right\} \tag{1}$$

where $p$ is the fluctuating pressure, $p_0$ is the ambient pressure, $S$ is the entropy, $u_i$ denotes the velocity components, $\gamma$ is the specific heat ratio, and $\mu$ is the coefficient of viscosity. This equation was first derived by Phillips.[1] For sound radiation processes in a turbulent flow, both heat conduction and viscosity are likely to be unimportant. Furthermore, if the flowfield is shock free, one can probably consider the effect of pressure fluctuation and the effect of entropy fluctuation separately. Under these circumtances, the last two terms on the right of Eq. (1) can be neglected.

In order to construct a solution to the convected wave equation, the flowfield is restricted to a parallel shear flow. This shear layer has a characteristic thickness of $L$, and its mean flow properties and the turbulent structure are homogeneous in the $x_1$ and $x_2$ directions. The mean flow velocity $\bar{u}$ and the local speed of sound $c$ are functions of $x_3$ only. Equation (1) can be further simplified by omitting small terms. In a turbulent flow, the fluctuating velocity components are small in comparison with the mean velocity. Terms depending on the small velocity fluctuations to the second or higher orders can be omitted. The simplified equation can be written as

$$\left\{\left(\frac{\partial}{\partial t}+\bar{u}_1\frac{\partial}{\partial x_1}\right)^2-\frac{\partial}{\partial x_i}c^2\frac{\partial}{\partial x_i}\right\}\log\left(\frac{p}{p_0}\right)=\gamma\left\{2\frac{\partial \bar{u}_1}{\partial x_2}\frac{\partial u_2'}{\partial x_1}+\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i}\right\} \tag{2}$$

where $u_i'$ denotes velocity fluctuations with zero mean.

In contrast to the Lighthill equation, the pressure fluctuation function appears only on the left hand side of the convected wave equation. Solutions to Eq. (2) are analytically complete. On the right-hand side of Eq. (2), there are two source terms. The first term is the shear noise, and the second term is the self noise. The inclusion of the self noise is a generalization from the corresponding equation of Ref. 1.

Since the parallel shear flow wave radiation problem can be regarded as statistically homogeneous in the $x_1$, $x_2$, and $t$ coordinates, Fourier transformation in these coordinates has been taken to reduce the convected wave equation to an ordinary differential equation of second order

$$\begin{aligned}(d^2/dy^2)\phi(y,\mathbf{k},\omega)+M^2q^2\phi(y,\mathbf{k},\omega)&=-M^2A^{-2}\Gamma(y,\mathbf{k},\omega)\\ q^2(y)&=A^{-2}(\omega+k_1V)^2-k^2/M^2\end{aligned} \tag{3}$$

where $\phi(y, \mathbf{k}, \omega)$ is the sound pressure fluctuation function $\Gamma(y, \mathbf{k}, \omega)$ is the source function, $M$ is the maximum speed ratio in the layer $U/c_0$, $A$ is the ratio of the local speed of sound to the ambient speed of sound $c/c_0$, and $V$ is a normalized velocity profile function $\bar{u}/U$. The definitions of $\phi$ and $\Gamma$ are given in the Appendix. All the coordinates are nondimensional: $y$ is the spatial coordinate in the transverse direction; $\mathbf{k}$ is the two-dimensional wavenumber in the plane of the shear layer; and $\omega$ is the frequency in the far field. This equation actually contains the Lighthill equation as a special case, where the velocity profile $V$ will be a

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finite step as $y$ goes from the moving medium to the ambient medium. Consequently, $q$ will also be discontinuous. Equation (3) can be solved completely by using a Green's function-integral equation approach. Solutions have been obtained for describing wave radiation processes in the subsonic, transonic, and supersonic convection velocity ranges. Three principal solutions are required for describing various important noise generating mechanisms in high-speed jets. These solutions consist of: the high-frequency Mach mode, designated as [S1]; the low-frequency Mach mode, designated as [S2]; and the acoustic mode, designated as [S0]. The general form of these solutions can be given as[2,3]

$$\phi(y, \mathbf{k}, \omega) = \psi'^{-1/2}\left\{H(\xi)+\int_0^{\xi} R(\xi, s)\,[g(s)H(s)+h(s)]\,ds\right\} \quad (4)$$

where $\xi$ is a transformed coordinate relative to $y$. The functions $\psi'$, $H(\xi)$, $R(\xi, s)$, $g(\xi)$, and $h(\xi)$ are different for each of the three modes, and their definitions are given in the Appendix.

Since the analysis of the generalized theory follows an approach which is different from the conventional methods employed in acoustics, the results are expressed in unfamiliar terms. It is necessary here to clarify the meaning of a few parameters.

The nondimensional frequency $\omega$, as defined in the fixed frame of reference, is directly proportional to the Strouhal number $S_t$

$$\omega/2\pi = fL/U = S_t \quad (5)$$

The function $q$ plays a central role in both the transformation of the wave equation and the final representation of the general solutions. In the shear layer, the value of $q^2$ can be positive, zero, or negative. In the far field, $q^2$ is a real and positive constant. From Eq. (3), one can find that $q$ is directly proportional to the wavenumber in the $y$ direction

$$k_3 = Mq \quad (6)$$

Hence, the wave propagation direction in the far field is determined completely by the three-dimensional vector $(k_1, k_2, k_3)$. The function $q$ also indicates the local wave propagation properties throughout the shear layer: the local wave motion is acoustical when $q$ is real; and the local wave motion, together with the local source function is hydrodynamical when $q$ is imaginary, $(q^2 < 0)$. When $q$ is real, $Mq$ does not exactly have the meaning of $k_3$ in the shear layer because $q$ is a function of $y$. However, Eq. (6) remains to be a good approximation for waves of very high frequencies. Some further discussion of the physical interpretation of $q$ can be found in Ref. 2.

The local frequency $\omega_0$ in the convected frame of reference is related to $\omega$ by the following equation

$$\omega = \omega_0 + k_1 V(y) \quad (7)$$

By knowing $q$, the chosen values of $k_1$, $k_2$, $\omega$, and $\omega_0$, one can define clearly the refracted path of wave propagation and wave velocity in the three-dimensional space.

The Doppler shift effect, which is explicit in the Lighthill formulation, is implicit in the generalized theory. However, the exact Doppler shift relation can be derived from Eqs. (6), (7), and the definition of $q$

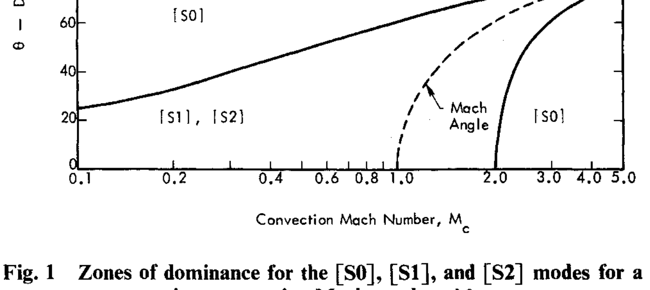


**Fig. 1 Zones of dominance for the [S0], [S1], and [S2] modes for a given convection Mach number, $M_c$.**

$$\omega = \omega_0/\{1-M_c\cos\theta\} \quad (8)$$

where $\theta$ is the angle between the $x_1$ axis and the direction of the far field wave propagation, and $M_c$ is the local convection Mach number, defined as $MV(y)$. It should be noted here that the refracted angle of propagation $\theta$ is fixed by the analytical solution for given $\omega$ and $k_1$. As a consequence, the refraction and convection effects are coupled. This coupled effect will be discussed later in this paper.

For the Mach modes, the principal sound source can be the frozen component of the turbulence in the moving coordinate. In this case, the far field sound frequency in the Mach modes is given directly by

$$\omega = k_1 M_c/M \quad (9a)$$

where $M_c$ is the convection Mach number at the layer containing the frozen turbulent structure. Furthermore, the direction of wave propagation will be given by

$$\cos\theta = M_c^{-1} \quad (9b)$$

The regions of dominance of the three modes of noise radiation are well defined. In [S0] mode, all the source function elements are locally acoustical, i.e., $\omega_0^2/k^2 = c^2$. The wave is being refracted and convected by the shear flow. When hydrodynamic pressure fluctuations in the turbulence are involved as source function, Eq. (3) can not be solved by elementary methods, and the WKBJ method must be employed. Solutions of the wave equation under such conditions are designated as [S1] and [S2]. In the [S1] and [S2] modes, the hydrodynamic pressure fluctuations, where $\omega_0^2/k^2 < c^2$, are convected and accelerated by shear flow to become an acoustic wave. The true Mach wave radiation, where $\omega_0^2/k^2 = 0$, is included as a special case of [S1] or [S2]. For a shear layer with a given convection Mach number, sound radiation from these modes are refracted into separate angular sectors. The computed results are given in Fig. 1. It is important to note that the [S1] and [S2] modes occupy a significantly large angular sector even for relatively small subsonic Mach numbers.

## Parametric Dependence of the Acoustic Intensity

It is possible to extract important parametric dependence relations from the analytic solutions, Eq. (3), by eliminating the quantitative details of the numerical values of various functions and integrals. This is an effective way to identify noise emission mechanisms in supersonic jet and rocket exhaust flows. Parametric dependence of the acoustic intensity for the three principal modes have been obtained†

$$[\mathrm{S0}] \quad \langle p^2(y, \mathbf{k}, \omega)\rangle \sim \gamma^2 p_0^{\,2}(c_j/c_0)^{-4} M_c^{\,4}(k/M)^2\Psi(y, \mathbf{k}, \omega) \quad (10)$$

$$[\mathrm{S1}] \quad \langle p^2(y, \mathbf{k}, \omega)\rangle \sim \gamma^2 p_0^{\,2}(c_j/c_0)^{-4} M_c^{\,5}\{(M_c+1)^2-1\}^{-0.5}\times (k/M)^{2.33}\Psi(y, \mathbf{k}, \omega) \quad (11)$$

$$[\mathrm{S2}] \quad \langle p^2(y, \mathbf{k}, \omega)\rangle \sim \gamma^2 p_0^{\,2}(c_j/c_0)^{-4}\Omega^{-0.5} M_c^{\,5}\times \{(M_c+1)^2-1\}^{-0.5}(k/M)^{2.50}\Psi(y, \mathbf{k}, \omega) \quad (12)$$

where $\langle p^2(y, \mathbf{k}, \omega)\rangle$ is the mean square fluctuating pressure intensity, $p_0$ is the ambient air pressure, $\Omega$ is a normalized shear gradient, and $\Psi$ is the spectrum of the fourth-order correlation of turbulent velocities.

The most important factors given in Eqs. (10–12) are the dependences of acoustic intensity on the convection Mach number and the frequency. In the nondimensional coordinates, the ambient speed of sound is represented by $M^{-1}$. Hence $(k/M)$ is directly proportional to the far field frequency $\omega$. The factor $(k/M)^{\nu}$ is very important for it governs the noise power dependence on convection. By substituting $\omega$ for $(k/M)$ in Eqs. (10–12), replacing $\omega_0$ for $\omega$ by means of Eq. (8), and incorporating a bandwidth adjustment

$$d\omega/d\omega_0 = (1-M_c\cos\theta)^{-1} \quad (13)$$

† The results given in Eqs. (10–12) are not the same as given in AIAA Paper 71-584 and Ref. 3. The derivation of these results, together with a complete documentation of the generalized theory, is currently under preparation for publication.

the parametric dependences can be written as

$$[S0]\quad \langle p^2(y,\mathbf{k},\omega)\rangle \sim \gamma^2 p_0^{\,2}(c_j/c_0)^{-4}M_c^{\,4}\omega_0^{\,2}\times (1-M_c\cos\theta)^{-3}\Psi(y,\mathbf{k},\omega_0) \tag{14}$$

$$[S1]\quad \langle p^2(y,\mathbf{k},\omega)\rangle \sim \gamma^2 p_0^{\,2}(c_j/c_0)^{-4}M_c^{\,5}\{(M_c+1)^2-1\}^{-0.5}\times \omega_0^{\,2.33}(1-M_c\cos\theta)^{-3.33}\Psi(y,\mathbf{k},\omega_0) \tag{15}$$

$$[S2]\quad \langle p^2(y,\mathbf{k},\omega)\rangle \sim \gamma^2 p_0^{\,2}(c_j/c_0)^{-4}M_c^{\,5}\{(M_c+1)^2-1\}^{-0.5}\times \Omega^{-0.5}\omega_0^{\,2.50}(1-M_c\cos\theta)^{-3.50}\Psi(y,\mathbf{k},\omega_0) \tag{16}$$

In a subsonic turbulent shear flow, the local characteristic frequency $\omega_0$ is proportional to the convection Mach number in the jet.[4] Hence, according to Eqs. (14–16), the over-all Mach number dependences in the low-subsonic Mach number range are $M^6$, $M^{6.833}$, and $M^{7.0}$ for the [S0], [S1], and [S2] modes, respectively. In the supersonic range, the Mach number dependence for all three modes will approach $M^3$. It is interesting to note that, in Eqs. (14–16), the factor $(c_j/c_0)^{-4}$ is equivalent to the dependence of jet noise on the density ratio $(\rho_j/\rho_0)^2$.

If a simple kinematically compatible turbulent structure is assumed, as adopted by Ribner,[5] and Pao and Lowson,[6] the spectral function $\Psi$ can be determined.[6] The broadband noise intensity dependences can be obtained by integrating Eqs. (14–16)

$$[S0]\quad \langle p^2(y)\rangle \sim M^6\{(1-M_c\cos\theta)^2+\alpha^2M_c^{\,2}\}^{-1.5} \tag{17}$$

$$[S1]\quad \langle p^2(y)\rangle \sim M^{7.33}\{(M_c+1)^2-1\}^{-0.5}\times \{(1-M_c\cos\theta)^2+\alpha^2M_c^{\,2}\}^{-1.667} \tag{18}$$

$$[S2]\quad \langle p^2(y)\rangle \sim M^{7.50}\{(M_c+1)^2-1\}^{-0.5}\times \{(1-M_c\cos\theta)^2+\alpha^2M_c^{\,2}\}^{-1.750} \tag{19}$$

where $\alpha$ is an assumed constant related to the turbulent structure.[6]

Other than the $U^3$-law in the supersonic Mach number range, many differences appear between the above results and the results derived from the Lighthill theory, which are discussed in detail by Ffowcs Williams,[7] and Ribner.[6] In the low Mach number range, the $U^8$-law is followed approximately, but not exactly, by the [S1] and [S2] modes. The [S0] mode follows a much lower power dependence, namely, $U^6$. However, the new results do not contradict experimental results. As previously discussed extensively in the literature,[4–7] the main portion of jet noise power is contributed by the hydrodynamic pressure fluctuation component of the turbulence because it contains more energy and the convection effect for such a source term is much stronger. Hence, the [S1] and [S2] modes may very well dominate the over-all acoustic power of subsonic jets. The jet noise radiation efficiency, as implied by Eqs. (17–19), has been computed for various Mach numbers, Fig. 2. A curve, representing the classical jet noise power dependence which was first given in Ref. 5, is also shown in Fig. 2 for comparison. It can be seen on Fig. 2 that the noise radiation efficiency for [S1] or [S2] mode agrees quite closely with the $U^8$-law for all subsonic convection Mach numbers.

Since rigorous analytical steps are followed in both the Lighthill theory and approaches taken by Phillips[1] and in the present study, it appears that the differences run deep into the mathematical structure on which various solutions have been constructed. Further studies in either the analytical techniques or the interpretation of results are necessary to resolve these differences.

The narrow band noise intensities in the far field, as given by Eqs. (14–16), depends on the ambient pressure, density of the jet flow, and the turbulence intensity. However, it does not depend on the integral spatial scale of the turbulence structure, since all variables containing such a scale are in nondimensional form. In the physical coordinates, both the frequency $\omega$ and the wave number $k$ depend on $a^{-1}$, where $a$ is the integral spatial scale of the turbulence. Hence, a factor of $a^{-1}$ is introduced into the broadband noise intensity dependence via the integration. In the initial region of the jet, the turbulent scale $a$, can be assumed to be proportional to $x_1$. The volume per unit length of the jet is also proportional to $x_1$. Therefore the sound source strength per unit length of the jet in this region is constant. In the developed region, the $x^{-7}$-law is recovered approximately.

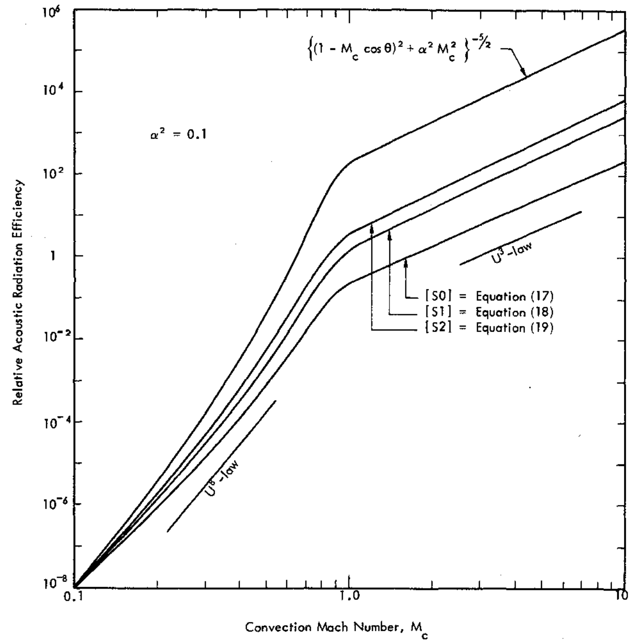


**Fig. 2 Jet noise radiation efficiency as a function of convection Mach number.**

This result agrees with the $x^0$-law and the $x^{-7}$-law as given by Ribner[8] for a subsonic jet. The agreement of the $x^0$-law with experiment has been verified. In the supersonic regime, the evidences are conflicting. In some detailed jet noise measurements obtained by Eldred et al.,[9] the source distribution in the initial region of the jet seem to follow the $x^0$-law very well. On the other hand, experiment given by Potter and Jones[10] and the recent results by Nagamatsu[11] indicate that the sound source distribution in the mixing region of the jet increases linearly with the distance from the jet nozzle, i.e., a $x^1$-law.

Although each segment of the jet produces broadband noise, the shape of the over-all spectrum is actually determined by the source strength distribution along the jet. The $x^{-7}$-law implies that the overall noise spectrum rises at about 6 db per octave. An $x^0$-law implies a fall off of 6 db per octave at the high-frequency end. This agrees with some rocket noise measurements by Cole[12] (Fig. 3). However, recent rocket noise and jet noise measurements indicate also that the fall off at high frequencies is close to 9 db per octave, which corresponds to a $x^1$-law of sound source strength distribution in the initial region of the jet. It does not seem easy to resolve this dilemma from a dimensional stand point.

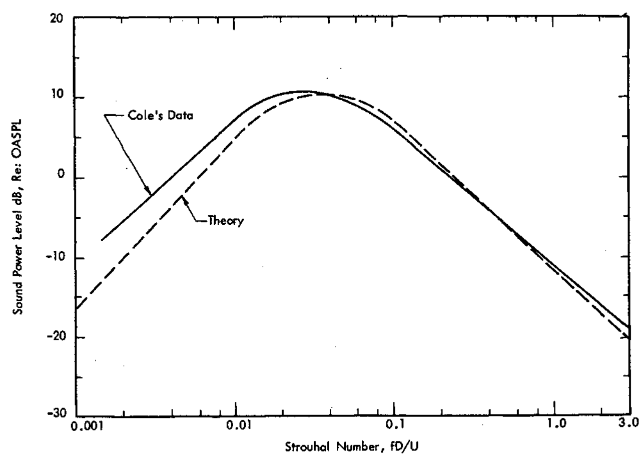


**Fig. 3 A preliminary prediction of rocket noise spectrum, (constant band width, 1 Hz).**

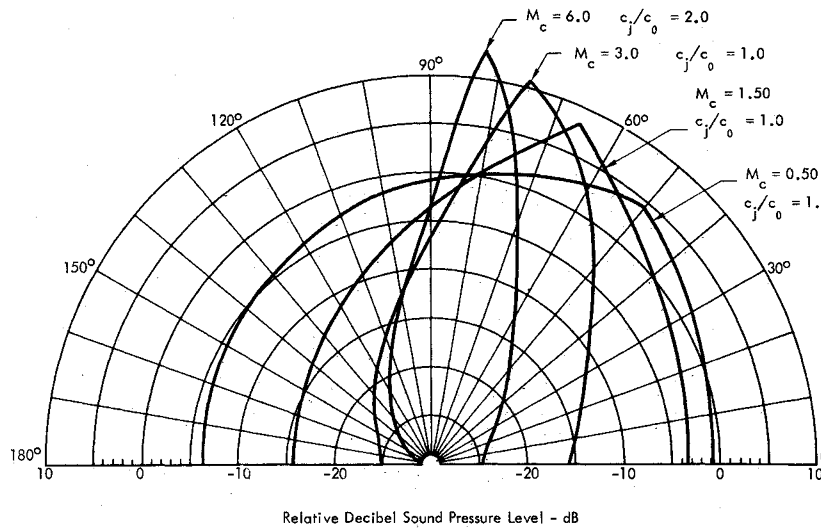


**Fig. 4 The normalized directivity factor produced by combined effects of refraction and simple Doppler shift.**

## Refraction and Convection

Since the generalized theory implicitly accounts for the effects of refraction and convection, the predicted noise intensity as a function of direction represents the true directivity pattern. Such a capability is of great importance for practical applications. The over-all directivity pattern is governed by a number of factors: convection, refraction, Doppler shift, the spectral characteristics of the turbulent source function, and the distribution of turbulent intensity in the jet exhaust flow. All of the above factors can be defined precisely by Eqs. (4–8). According to the analytical structure of Eq. (4), the noise intensity in a given direction and at a given frequency is an integral of all source contribution along the wave propagation path.[2] Hence, even for simple given velocity and turbulence structure profiles of the shear flow, detailed numerical computations are necessary to construct a complete directivity pattern in the far field. Nevertheless, some essential refraction and convection properties can still be extracted from the general solution.

It is interesting to find through the analysis of the generalized theory that the refraction and the Doppler shift effects are independent of the detail structure of the shear flow. These effects depend only on the convection Mach number $M_c$ and the speed of sound ratio $A$. The Doppler shift effect has been given by Eq. (8). For given values of $M_c$ and $A$, noise radiations from the [S0] mode and the [S1], [S2] modes are refracted into different angular sectors in the far field.

Both the Mach modes, [S1] and [S2], will radiate into an angular sector between $\theta_1$ and $\theta_2$ where

$$\cos\theta_1 = (M_c - A)^{-1}, \qquad \cos\theta_2 = (M_c + A)^{-1} \tag{20}$$

and $\theta_1 = 0°$ if $(M_c - A) \leqq 1$. These limiting angles have been given in Fig. 1 for $A = 1.0$. For the same given values of $M_c$ and $A$, the [S0] mode will radiate into two angular sectors

$$0° \leqq \theta < \theta_1 \qquad \text{and} \qquad \theta_2 < \theta \leqq 180° \tag{21}$$

Since the sound source element in the [S0] mode is locally acoustical, it has a definite local wave propagation drection $\theta_0$. The relation being $\theta_0$ and the corresponding propagation angle in the far field $\theta$ is given by

$$\cos\theta = \cos\theta_0 (M_c \cos\theta_0 + A)^{-1} \tag{22}$$

For the [S1], [S2] modes, the source element is locally hydrodynamical. Only the components with a real wavenumber parallel to the shear layer can radiate sound into the far field. Hydrodynamical sources oriented in other directions will radiate decay waves which will reach only the near field. The far field radiation angle, $\theta$, is governed here by the local phase speed—$\omega_0/k$‡

$$\cos\theta = (M_c - M\omega_0/k)^{-1} \tag{23}$$

The refraction effects of the acoustic mode has been discussed before in the literature. The radiation of the [S1], [S2] modes has been overlooked before because it falls into the so-called shadow zone.

For a given constant far field sound frequency, the wave energy contained in the solid angle element between $\theta$ and $\theta + d\theta$ will correspond to an [S0] mode acoustic source element located between $\theta_0$ and $\theta_0 + d\theta_0$, or a Mach mode source element bounded by $k_1$ and $k_1 + dk_1$ in the wavenumber frequency space. If the source function is assumed to have equal energy density per unit volume in the wavenumber frequency space, then the far field noise energy density will not remain uniform because volume element has changed in size. The intensification or rarefaction of sound intensity owing to refraction can be obtained. Furthermore, the simple Doppler shift effect will also modify the wave intensity by an additional factor of $(M_c \cos\theta - 1)^{-1}$. The combined effects of refraction and Doppler shift on directivity can be given by the following formulas:

$$\begin{aligned} f(\theta) &= A(1 - M_c \cos\phi)^{-3} \quad 0 \leqq \theta < \theta_1;\ \theta_2 < \theta \leqq 180° \\ f(\theta) &= A^{-2}(M_c - M\omega_0/k)^3 \quad \theta_1 \leqq \theta \leqq \theta_2 \end{aligned} \tag{24}$$

The directional factor, as given by Eq. (24) has been computed and normalized for several Mach numbers, Fig. 4. It should be noted that at the Mach angle the amplification factor of sound radiation remains finite. The infinite Doppler shift factor is compensated for by a zero width of the source volume.

In the generalized theory, the convection factors owing to multipole effect is represented in terms of $(k/M)^v$, as given by Eqs. (10–12). The directivity factor produced by the multipole convection effect remains finite for all angles, and it applies to the far field noise in addition to the factor as given by Eq. (24). One may recall that the apparent singularity produced by the $(1 - M_c \cos\theta)^{-5}$ factor in the Lighthill theory can be alluviated by chosing $k$ as an independent spectral variable instead of $\omega$ (Ref. 6).

For [S1] and [S2] modes, Eq. (4) describes the wave radiation in a fixed direction[2] $\theta$. A coordinate transformation $\xi$ is chosen such that the origin of $\xi$ corresponds to a depth in the shear layer $y_r$. For $y \leqq y_r$, hydrodynamic sources will be contributing to noise radiation along the wave path, while, for $y > y_r$, the acoustic sources will be contributing. The principal sound source for [S1] and [S2] modes is located at depths of $y \leqq y_r$ because radiation from sources in the $y > y_r$ layer is equivalent to the [S0] mode radiation.[3] If the convection Mach number at $y_r$ is designated as $M_r$, then the far field sound frequency will be

$$\omega(\theta) = k_1(1 + M_r)/M; \qquad \cos\theta = (M_r + 1)^{-1} \tag{25}$$

This formula is actually equivalent to Eq. (9). According to the structure of Eq. (7), both $\omega$ and $k_1$ are fixed. The local frequency

‡ Because of the chosen definition of the Fourier transform, a forward propagating wave is signified by a negative ratio of frequency to wavenumber.

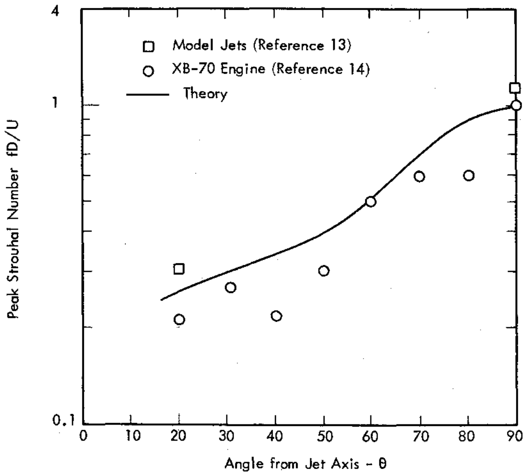


**Fig. 5 Peak Strouhal number vs direction for jet noise in the far field.**

$\omega_0$ varies with $y$. Using this equation a hypothetical frozen turbulence layer can always be defined such that the local turbulent frequency vanishes. If such a Mach number is designated as $M_c$, then

$$M_c = M_r + 1 \tag{26}$$

By substituting Eq. (26) into Eq. (25), the result is the same as Eq. (9).

One of the important aspects of jet noise radiation is the variation of peak Strouhal number as a function of direction. This effect has been observed in experiments. It is feasible here to predict the general trend of peak Strouhal number as a function of $\theta$ using Eq. (25). It was pointed out earlier that it is plausible that the overall noise level may be dominated by the [S1] and [S2] mode. For subsonic jets, the turbulent structure is similar. The characteristic wave number can be estimated.[5] Hence, Eq. (25) can be used to predict the peak Strouhal number for angles up to $\theta_2$, as defined in Eq. (20). Beyond this angle, estimations can be made based on the self noise radiation by the same characteristic wavenumber in the turbulence. The results of this computation are shown in Fig. 5.

The prediction is compared to some jet noise measurements. The first set of data is given by Lowson.[13] A model jet of 2.84-in. diam was used, and the exit velocity ranges from 400 fps to 1000 fps. The second set of data was reported by Lasagna and Putnam[14] on measurement of XB-70 engine noise in the far field. The exit Mach number was approximately 1.9 and the exit velocities range from 2000 to 3000 fps. Fair agreement between the theory and the data points is observed. The XB-70 noise data is generally lower than the predicted curve. This is probably a temperature effect as discussed before by Rollin.[15] For a high-temperature jet, the spread of the jet was observed to be 30% wider than a cold jet. Consequently, the characteristic frequencies will be lower by the same percentage.

In the conventional treatment of jet noise, refraction and convection are often considered separately, and the effects are applied in sequence. However, the generalized theory indicates that the refraction and convection effects are coupled. As a consequence, the convection effect is significantly weakened by the refraction effect for large convection Mach numbers.

In the generalized theory, the convection effect is associated with only the longitudinal wave number. In the hydrodynamic pressure fluctuation regime, the convection effect increases the local frequency of the longitudinal wave components until the phase speed finally reaches the local speed of sound. This longitudinal hydrodynamic pressure fluctuation becomes a longitudinal wave component. While the frequency is being increased further by the convection, the transverse wave number $Mq$ begins to grow. The three-dimensional wave number maintains always a magnitude such that the frequency equals the product of the wave number and the local speed of sound, as required by fundamental fluid dynamical principles. Through this process, the convection factor is weakened. For example, a local wave component with frequency $\omega_0$ in the turbulence, convected at a Mach number $M_c$ parallel to the wave propagation direction, will have a Doppler shift of

$$\omega = \omega_0/(1-M_c); \qquad M_c < 1 \tag{27}$$

The subsonic Mach number is chosen mainly for the simplicity of argument, and the generality of the physical phenomenon is not affected.

By including the refraction effect, the Doppler shift will be

$$\omega = \omega_0/(1-M_c\cos\theta) = \omega_0\{1+M_c\}$$

with

$$\cos\theta = \{1+M_c\}^{-1} \tag{28}$$

By comparing Eqs. (27) and (28), it is interesting to note that they are approximately equal for small convection Mach numbers. For the high subsonic Mach numbers, the Doppler shift given by Eq. (27) is much larger than the Doppler shift of Eq. (28). The leading error term in Eq. (27) is proportional to $M_c^{\,2}$, which has the same dimensional dependence as the compressibility effect. The compressibility effect is indeed an underlying mechanism which modifies the classical Doppler shift law.

## Peak Strouhal Number for the Over-All Noise Spectrum

For a supersonic jet flow, such as a rocket exhaust, the Mach mode radiation dominates the power spectrum. The sound frequencies in this regime depends on the longitudinal wave number spectrum of the frozen turbulence component in the jet flow, and not on the time rate of change of the turbulence. In nondimensional coordinates, the integral spatial scale of the turbulence is proportional to $M^{-1}$. In other words, the correlation distance in the transverse direction is proportional to the distance between two points where the difference of convection speed is equal to the local speed of sound. There have been very few experimental results regarding turbulent structures in supersonic flows.

Phillips[16] indicated that the ratio of the longitudinal integral scale to the transverse turbulence scale can be as large as 18:1. In some measurements in a Mach 3.0 supersonic wake, Demetriades[17] found that the ratio is 10:1. According to Townsend,[18] this ratio is only 3:1 for low-speed shear layers. For noise prediction purposes, it is very important to know this scale ratio. For the above given numbers one can either assume that the ratio is a constant for all supersonic flows, or that the ratio increases with Mach numbers. The latter case is plausible because the potential core length of a supersonic jet does increase linearly with $M$ (Fig. 6).

The spreading rate of the jet exhaust flow in the transition

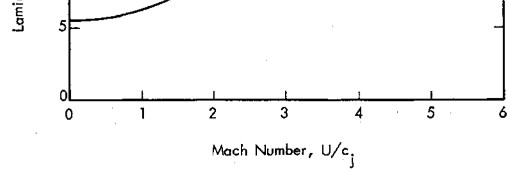


**Fig. 6 The laminar core length as a function of jet exhaust Mach number.**

zone is relatively small. The characteristic dimension can be chosen as $D$, the exit diameter of the jet under perfect expansion conditions. According to the generalized theory, the far field noise frequency $\omega$ is related to the longitudinal wave number $k$ and the convected Mach number $M_c$, via Eq. (9).

If the transverse correlation length in the turbulence is assumed to be

$$L_y = A/M$$

the characteristic transverse wave number (peak) for the [S1] or [S2] modes of radiation can be given as

$$(k_3 A/M) = \pi$$

It is plausible to define the longitudinal wave number by an empirical formula

$$(k_1 A/M) = 0.30\pi/M \tag{29}$$

The scale ratio is assumed to be proportional to the Mach number. Equation (29) gives a scale ratio of 3.3 for $M = 1$ and a scale ratio of 10 for $M = 3.0$. Hence, the peak Strouhal number will be

$$S_t = \omega/2\pi = (k_1/2\pi)(M_c/M) = (0.15/A)(M_c/M) \tag{30}$$

If $(M_c/M)$ is assumed to be 0.60 and $A = 3.0$, then the peak Strouhal number will be

$$S_t \simeq 0.03$$

This estimated peak Strouhal number is somewhat higher than the peak Strouhal number commonly measured in rocket noise power spectrum. The discrepancy can be the direct result of an underestimate of the longitudinal spatial scale of the turbulence. It is more likely, however, that the source volume corresponding to the peak frequency is actually located futher downstream. Since both the wave number and the convection speed are lower, the peak Strouhal number for the over-all power spectrum will also have a lower estimated value.

It is interesting to note from Eq. (30) that the peak Strouhal number for supersonic jets is inversely proportional to the speed of sound in the jet. This relation suggests that a modified Strouhal number

$$S_t = (fD/U)\, c_j/c_0$$

can be a good parameter for correlating supersonic jet noise.

The same approach can be applied for estimating the peak Strouhal number for the over-all noise spectrum for transonic jet exhaust flows. From experimental evidences, the aerodynamic structure of all subsonic round jets are essentially similar. The laminar core length, turbulent mixing, and turbulence scale remain invariant for all subsonic Mach numbers. According to the generalized theory, the forward noise radiation in subsonic jets are also governed by [S1] and [S2] modes. The angle $\phi_{max}$ is 25° for a convection Mach number as low as 0.1. In the high subsonic range, the over-all noise power is definitely governed by Mach modes. Since the transverse turbulence scale is proportional to the width of the jet, longitudinal turbulence scale is approximately three times the transverse turbulence scale, the characteristic longitudinal wave number can be given as

$$k_1 = 0.33$$

By using Eq. (22), it follows that the peak Strouhal number is:

$$S_t = 0.167(1 + M_c)/M \quad \text{for} \quad M_c \leqq 1 \tag{31}$$

If $M_c/M = 0.60$, then $S_t$ will take a value between 0.28 and 0.20. It is not unreasonable to find that the peak Strouhal number as given by Eq. (31) is too large for the lower convection Mach numbers. In this range, the over-all noise spectrum is actually being dominated by the space-time structure of the jet turbulence controls the sound radiation process. In a previous study of subsonic jet noise by Pao and Lowson,[5] an accurate estimate of the peak Strouhal number was obtained through using the Lighthill theory.

The estimated peak Strouhal number of jet noise spectrum is shown in Fig. 7. In the generalized theory of jet noise, it appears that the noise spectral characteristics relate directly with the jet structure, which is in turn related to the Mach number. Hence,

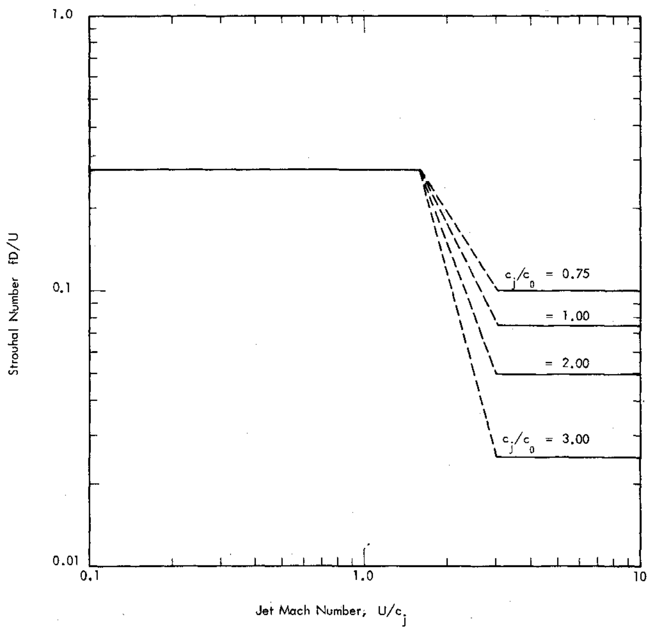


**Fig. 7 Peak Strouhal number for the over-all jet noise spectrum.**

the peak Strouhal number in Fig. 7 is plotted against the actual jet Mach number intead of the jet exit velocity.

The prediction of spectral characteristics of jet noise in the low supersonic range would require detailed numerical calculations because the dominance of various mechanisms is unknown through a simple parameter study. Hence, the predictions of Strouhal number in the subsonic regime and the high-supersonic regime are connected by dash lines. Currently, both analytical and numerical studies of these unknown properties are in progress.

## Appendix

For Eq. (3), the following functions have been defined:

$$\phi(\mathbf{y}, \tau) = A(y_3)\log(p/p_0); \quad G(\mathbf{y}, \tau) = \gamma A(y_3)(\partial v_i/\partial y_j)(\partial v_j/\partial y_i)$$

$$\phi(y_3, \mathbf{k}, \omega) = (2\pi)^{-3} \iint \phi(\mathbf{y}, \tau) \exp -i(\mathbf{k}\cdot\mathbf{y} + \omega\tau)\, d\mathbf{k}\, d\tau$$

$$\Gamma(y_3, \mathbf{k}, \omega) = (2\pi)^{-3} \iint G(\mathbf{y}, \tau) \exp -i(\mathbf{k}\cdot\mathbf{y} + \omega\tau)\, d\mathbf{k}\, d\tau$$

where $y$, $\tau$, and $v_i$ are nondimensional variables:

$$y_i = x_i/L, \qquad \tau = tU/L, \qquad \text{and} \qquad v_i = u_i/U$$

In Eq. (4), $\xi$ is a coordinate transformation, $h(\xi)$ is a transformed source function, $H(\xi)$ is an arbitrary linear function of the two independent solutions to the homogeneous wave equation, $R(\xi, s)$ is a resolvent kernel corresponding to the kernel function $K(\xi, s)$ in an integral equation equivalent to Eq. (3), $g(\xi)$ is an error function introduced by the WKBJ transformation, and $\psi'$ is the first derivative of $\xi$ with respect to $y$. Other than the general definition of $h(\xi)$ and $g(\xi)$

$$h(\xi) = -(\psi')^{-1.5} A^{-2} M^2 \Gamma(\xi, \mathbf{k}, \omega)$$

$$g(\xi) = \tfrac{1}{2}(\psi')^{-2}\{\psi'''/\psi' - \tfrac{3}{2}(\psi''/\psi')^2\}$$

all other functions are different for each mode. These functions are given below for each of the three modes separately. For [S0]

$$\xi = \int^y q(y)\, dy; \quad \psi' = q(y); \quad g(\xi) = 0$$

$$H(\xi) = a_1 \cos M\xi + a_2 \sin M\xi$$

$$K(\xi, s) = M^{-1}\{\sin M\xi \cos Ms - \cos M\xi \sin Ms\}; R(\xi, s) = K(\xi, s)$$

In the preceding formulas, $a_1$ and $a_2$ are arbitrary constants to be determined by using proper boundary conditions. For [S1]

$$\xi = \left\{\frac{3}{2}\int_{y_r}^{y} q(y)\,dy\right\}^{2/3}; \qquad \psi' = q(y)/\xi^{1/2}$$

$$\psi' = (k/M)^{2/3}(2M\Omega k_1/k)^{1/3} \qquad \text{as} \qquad \xi \to 0$$

$$H(\xi) = a_1\, Ai(M^{2/3}\xi) + a_2\, Bi(M^{2/3}\xi)$$

$$K(\xi,s) = \pi M^{-2/3}\{Ai(M^{2/3}\xi)Bi(M^{2/3}s) - Bi(M^{2/3}\xi)Ai(M^{2/3}s)\}$$

$$R(\xi,s) = K(\xi,s) \quad \text{as a first approximation}$$

In the preceding formulas, $Ai$ and $Bi$ are the Airy functions, and $\Omega$ is the nondimensional mean velocity gradient. For [S2]

$$\xi = \left\{2\int_{y_r}^{y} q(y)\,dy\right\}^{1/2}; \qquad \psi' = q(y)/\xi$$

$$\psi' = (k/M)^{1/2}(M\Omega k_1/k)^{1/2} \qquad \text{as} \qquad \xi \to 0$$

$$H(\xi) = a_1\, Pa(M^{1/2}\xi) + a_2\, Qa(M^{1/2}\xi)$$

$$K(\xi,s) = \pi M^{-1/2}\{Pa(M^{1/2}\xi)Qa(M^{1/2}s) - Qa(M^{1/2}\xi)Pa(M^{1/2}s)\}$$

$$R(\xi,s) = K(\xi,s) \quad \text{as a first approximation}$$

In the preceding formulas, $Pa$ and $Qa$ are functions related to the Bessel functions of one-fourth order

$$Pa(\xi) = 2^{-1.5}\xi^{0.5}\{J_{-1/4}(\tfrac{1}{2}\xi^2) - \alpha J_{1/4}(\tfrac{1}{2}\xi^2)\}$$

$$Qa(\xi) = \tfrac{1}{2}\xi^{0.5}\{J_{-1/4}(\tfrac{1}{2}\xi^2) + \alpha J_{1/4}(\tfrac{1}{2}\xi^2)\}$$

where $\alpha$ is a complex constant satisfying $\alpha^4 = -1$.

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